The Impact of Structural Algebraic Units on Students' Algebraic Thinking in a DGS Environment

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Abstract: *The present paper attempts to bridge the world of digital technology and the world Euclid bequeathed us in his "Elements". The role of the design process of activities in an dynamic geometry interactive environment such as that of Geometer's Sketchpad v4 is examined, along with ways in which students can be assisted to understand algebraic concepts through geometrical reconstructions. The ways in which it can facilitate the understanding of geometrical concepts are examined, along with the bridging, between the fields of algebra and geometry, and strategies for overcoming obstacles. Two volunteer teams (one control, one experimental) were evaluated with regard to their ability to represent concrete algebraic expressions using geometrical representations on cardboard. From the results, it can be concluded that the experimental team managed not only to construct the concrete identities, but also to connect them with formal reasoning.*

1. Introduction

The geometrical representations of identities and algebraic expressions in general were initially developed in Euclid's "Elements" (325-265). Stamatis [14] declares: "The second book of Euclid's "Elements"… includes the application of Geometry to Algebra and is ascribed for the most to Pythagoreans. The first 10 theorems relate to algebraic identities, which we are able to represent in the following way: with the letters a, b, c assigned to represent straight segments." (my translation from the Greek text). Netz [28] also shows that the Greeks considered diagrams essential to geometrical proofs and they allowed properties to be conceptualized from the diagram. Fowler [7] makes it clear Greek geometry was non-arithmetical and did not use fixed units for measurement [10]. Herbst's opinion is [12] that Netz's [28] study of lettering practices in Greek geometry permits the observation that Greek geometers produced their diagrams at the same time that they conceived their proofs: i.e. the diagram was not drawn at the end to illustrate the written proof, and was not drawn in its entirety before the production of the argument".

Representations were the first empirical mode leading to the proving process in Ancient Greece, (see, for example, Socrates and Meno) although the process observed in Euclid's "Elements" does not display a transition from visual representation to rigorous reasoning. The visual representations can prove only specialized cases, while the Euclidean proof can empower every case by reinforcing the initial visual proof. In the Platonic dialogue (Socrates: slave), the slave's incorrect answers are restructured with a shape: a concrete representation, with the particular shape functioning as the visual proof of the accuracy of Socrates' proposition to the slave. The phrase "ei mi voulei arithmein alla deixon" which means "if you don't want to measure, just prove" (my translation from the ancient Greek text) can be considered an interpretation of what the ancient Greeks meant by 'proof', making use of the concrete expression of the ancient verb "deiknymi".

Furinghetti and Paola [16] argue that the didactical suggestion implicit in Lakatos' work is that a return to the spirit of the Greek geometers would be advisable. Referencing Szabo, they call for 'deiknymi' to be developed both analytically and synthetically [16]. How meaningful can the act of "deiknimi" be nowadays, and how do students understand it? How can this be achieved through modelling such as composing and decomposing of shapes [32])? How important is the mediation of the tools or the materials or the artifacts for students to construct knowledge?

Researchers around the world concur in the view that greater emphasis should be put on activities in software that actively involve students (see for instance [8]). Designing learning activities, and making them effective in their static and dynamic dimensions, is a complex pedagogical task that lies at the heart of teacher practices. Moreover, the sloppiness and inaccuracy of man-made constructions can also be avoided through the use of technology. As Mariotti [26] argues, the effect is powerful enough to legitimately talk of "a new experiential mathematical realism".

This paper discusses a dynamic geometry environment by which students understand algebraic concepts. Extending the paper's aims, we could say it relates to learning in Mathematics and to the applications of technology to mathematical education. The issues that will be addressed in the paper are the following: a) section 3 will deal with the constructivist approach underlying the design of the activities in view of the learning theories mentioned in section 2; b) an evaluation of two control teams with regard to their ability to represent algebraic expressions using geometric constructions.

2. Decomposing Euclid's proposition using *structural algebraic units*

The first ten propositions of Book II can be easily interpreted in modern algebraic notation, and for this reason the subject matter of Book II is usually called "geometric algebra". The proposition 4 of Euclid's Elements (Book II) supports that: "If a straight line is cut at random, then the square on the whole equals the sum of the squares on the segments plus twice the rectangle contained by the segments." One may of course interpret proposition 4 in algebraic terms, but strictly speaking one cannot say that Euclid developed "geometrical representations of identities and algebraic expressions"; it would be much better to say that "after Euclid mathematicians developed algebraic representations of his geometrical propositions", and by so doing they extended the domain of application to any kind of numbers (and consequently their meaning): for instance the identity $(a + b)^2 = a^2 + 2ab + b^2$ is also valid when a and/or b are negative numbers (or even complex numbers).

For instance in Euclid's proof of proposition 4 (book II) numbers do not appear as such (and even less algebraic expressions), since this proposition deals only with lengths. It is not just a question of translating into "modern notation" but a question of what Euclid had in mind. Nowadays, the historical process is reversed since in classrooms geometrical representations are used to introduce or illustrate algebraic formulas. The statement of the proposition can be interpreted in modern algebraic notation as saying that if $x = a + b$, then $x^2 = a^2 + b^2 + 2ab$ or as an identity, it says $(a + b)^2 = a^2 + b^2 + 2ab$. In fact, this modern interpretation of Euclid's statements should be put in the correct perspective. I would not try to show that this interpretation can be useful and effective because it is the 'original', but the opposite because this is the way that it can be interpreted nowadays.

Working backwards from Euclid's method, we see that the terms a^2, ab can be translated into areas by composing the initial shape out of its component parts or, in other words, by modelling the algebraic expression geometrically. Rahim in [32] writes "mathematically and in a Euclidean sense, this combination allows us to extend the concept of congruence from rigid shapes congruence (same intact shapes) to congruence by pieces (different shapes) thus opening a wider range of interesting possibilities". At the same time, these areas can also serve as algebraic units (as can, for instance, the terms x^2, x, 1), allowing the form of an identity —or, more generally, a polynomial—to be constructed (or structuring an identity or polynomial). They are, in other words, '*structural algebraic units*'. These algebraic units are only meaningful when they are given substance, shape and form during the learning and teaching of algebra; it is generally accepted that the use of materials and tools during the learning process can help the learner visualize the meanings involved.

The artifacts used to model or represent mathematical processes of concepts are made out of materials constructed by man, either they be palpable (a card construction, for example) or digital (a dynamic geometry software construction). In that sense the abstract meanings are rendered concrete by means of the figurative /schematic material. This is not a specificity of the DGS tool, but can be achieved as well through cutting up paper shapes. For example the students could be helped by the constructions of the basic shapes, and this could be the same if they had constructed them with instruments on a sheet of cardboard before cutting them up. But the representation of structural algebraic units using sheet of cardboard is a time-consuming process, and is not recommended for the construction of algebraic constructions in class, at least with High School students. In this sense, Euclid's propositions become meaningful for the active teaching and learning of algebra when they are presented as constructional activities and open problems whose structural units have to be assembled and reassembled in a dynamic geometry environment. Taking this approach, students acquire an interest in the history of mathematics, understand that algebraic expressions of identities are meaningful in reality because they are representative of the areas of shapes, and understand the area of a complex geometric shape to comprise of the areas of the component shapes of which it is composed. This aspect will be examined below. For the design of activities, I took into account the theories referred to in the next section relating to knowledge, learning and teaching during the design and implementation of the activities.

3. Theoretical framework

Mathematics visualization and connections, links and relations between representations have appeared in recent literature as fundamental aspects to understanding students' construction of mathematical concepts, as well as important characteristics of learning and problem solving (see for example [18] and [21]). The constructivist view of representation as conceptual knowledge is consistent with the notion that learners actively construct new knowledge in problem solving situations "when their current knowledge results in obstacles, contradictions, or surprises" ([2] p. 92). The subject of this study is clearly linked with the notion of semiotic register developed by Duval [3]. The semiotic registers used in the mathematical activities are the algebraic, the graphical, the figurative and the natural language. A semiotic register, according to Duval [3], constitutes a system of representation if it allows three cognitive fundamental characteristics: its production, a treatment, and a conversion between different semiotic registers. Thus [4], the operative connections we expect during learning differ in their registers of semiotic representation and not between deductive and empirical mathematics, proofs and constructions or mathematical and symbol structures.

Another possible point of view could have been to refer to conceptual metaphors (see for instance [13]). In the present case, a geometrical diagram can be considered as a metaphor for the corresponding algebraic formula. Sfard [36] draws on the work of Lakoff and Johnson, reporting that a metaphor is "a mental construction which plays a constitutive role, in structuring our experience and in shaping our imagination and reasoning" (p. 46).

Kaput, Noss, and Hoyles in [23] analysed *new representational infrastructures*, namely "the ways we use to present and re-present our thoughts to ourselves and to others, (in order) to create and communicate records across space and time, and to support reasoning and computation" and how "the associated artifacts and technologies have, over long periods of time, gradually externalized aspects of knowledge and transformational skill that previously existed only in the minds and practices" [23]. The interaction with representations in a computing environment has two aspects: "the action upon a representation by the user through the intermediary of a human-computer interface, and the representation communicating back through some form of reaction or response" [34]. Kadijevich & Haapasalo [19] argue that, using computers, students can spend less time on procedural skills and more on developing their conceptual understanding [5]. Given the core role in mathematics education of developing procedural and conceptual knowledge and forging links between the two, a key research question is how different technologies affect the relationship between the two.

The theoretical framework includes the notions of instrumental genesis [41] and the distinction between phases of instrumentation and instrumentalization [9], which are fundamental in teaching in computer environment. Adapting the term of Trouche [40], Haapasalo [11] *instrumentation* means *the* process when the tool shapes the actions of the users. On the other hand, users often find their own schemas and schemes to utilize the tool. In this process of *instrumentalization* not only the usage of the tool, but also the objects to be investigated are shaped by the users" [11].

During the instrumental genesis both the phases (instrumentation and instrumentalization) coexist and interact. Then the user structures what Rabardel [31] calls utilization schemes of the tool/artifact. Utilization schemes are the mental schemes that organize the activity through the tool/artifact. This process involves many studies, among them, for example the one of Artigue [1], based on the research of Verillon & Rabardel [41] about the ways by which an artifact becomes an instrument for a student. According to Artigue [1], "An instrument is thus seen as a mixed entity, constituted on the one hand of an artifact and, on the other hand, of the schemes that make it an instrument for a specific person. These schemes result from personal constructions but also from the appropriation of socially pre-existing schemes." One must also bear in mind the teacher's role, as introduced by Mariotti [26], in this approach: it can in no way be considered certain that an artifact will function in the construction of this meaning, even if it does incorporate a mathematical meaning and schemes of use. That said, an artifact can encourage students to fully utilize their communication strategies and guide them towards the desired meaning if it is placed between the teacher and the students.

In the following section, we shall examine the reasons why the proposed constructional processes which make use of software tools are superior to the ways in which they can be constructed in a static environment. The advantages of the former, in terms of the way in which students construct knowledge, are: a) whether they themselves are operating the software when constructing shapes like those described in subsection 4.1 using the tools provided in the menu or b) working on diagrams (semi) predesigned by the teacher and activated by the students themselves using interactive techniques like those described in subsection 4.2 which can facilitate students guiding them towards the correct way to link the sequential constructional steps and the visual proof of the problem. These processes are presented in the next section using two problems which are reworkings of propotitions 4 & 5 in Book II of Euclid's Elements.

4. The Constructivist approach

4.1. Composing the shapes

Problem1: Construct 4 shapes: *a square of side a, a square of side b and two rectangles in different colours with sides a, b. Then construct a larger shape matching the shapes which you have already constructed in order to construct a quadrilateral.* (GSP *file page 1*)

Working using static means, we can use a compass and a straightedge (ruler with measurements). Dynamic geometry systems were designed to facilitate the teaching and learning of Euclidean geometry. Dynamic geometry systems such as the Geometer's Sketchpad [17] or Cabri II [24] (or any other DGS software) are microworlds, and as such "even if the programs differ in their conceptual and ergonomic design, they share: a) a dynamic model of Euclidean Geometry and its tools (the dragmode); b) the ability to group a sequence of construction commands into a new command (macro-constructions); c) the visualisation of the trace of points which move depending on the movement of other points (locus of points)” [38].

The basic tools of a dynamic geometry environment are a) Circle (equivalent to Compass) b) Segment/Ray/Line (equivalent to Unmarked Straight Edge) c) Point (which simply enables us to place one of the fundamental 'objects' of Euclidean geometry) d) Pointer (which crucially enables us to drag objects) [25]. When these tools are combined with the software’s options menu, they allow the user to produce constructions which must conform with the principles of Euclidean geometry if they are to function and pass the dragging test. This means that the student has to know the theory of geometry if s/he is to generate a correct geometric construction. And while we have explained that, in the software, the constructions can contain the same mathematical logic as the constructions on paper, there are substantial differences in the manner in which the tools are used.

Mariotti [27] declares that in a construction generated using dynamic geometry software “the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality”. In order to comprehend the advantages (and disadvantages) of the construction mode in the dynamic geometry software, it is necessary to examine the differences between it and the mode of construction using static means. This will allow us to compare the two modes. For instance, in using a straightedge with measurements, the mode of constructing a figure in the software (e.g a square of side a) could be different from the mode students use to construct it on paper. For example: When a pupil works using static means, s/he is able to measure the length of side ‘a’ with a ruler. Afterwards s/he is able to use this measurement to construct either the next side of the square given that s/he knows the geometrical properties of the shape or of another shape (for example a rectangle) whose one side is equal to a. This “measurement” method is not the method the software demontrates for constructing a shape which does not mean that the software faces a constructional inability/disadvantage comparing it with static means. According to Duval [4] “We can have even a conflict between the figure and the measures leading to a paradox”.

Duval (ibid.) argues that “Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization.” By forcing students to think of ways of constructing an equal segment, this methodological weakness can thus provoke a cognitive conflict in students, and in so doing raise the level of difficulty. One such way would be to define side ‘a’ as an arbitrary segment on the screen and then use it as a radius of a circle in the construction. This construction method induces a different mental perception in the students with regard to construction in the software.

In this way, the sides of the square cannot be modified from the vertices of the shape using the dragging modality. Instead, they depend on the modification of the initially defined segment a. The arbitrary segment ‘a’ could thus be confined as a non-collapsible compass to either the square or any shape whose a side is equal to ‘a’. This construction procedure depends on the students’ level of conceptual knowledge and cognitive abilities. In Fig. 4.1.1, the hidden elements have appeared in the construction and we are able to demonstrate the construction process of the shapes. As a consequence, the construction of the shapes depends either on segments a & b--both of which are arbitrary defined --and the relationship among them, or on the students’ geometrical knowledge of the relationships between and properties of shapes.

Another important point about this construction is that students can use it to verify and to construct arguments, and in so doing overcome an epistemological obstacle which one frequently finds obscuring comprehension in pupils taking tests using static means, and in which they mistakenly write that $(a+b)^2 = a^2+b^2$. Obstacles like this can be seen as an opportunity for students to reflect on their own learning rather than allow this to be a barrier to achieve understanding of mathematical ideas. Through the software constructions the abstract meanings will be rendered concrete by means of the schematic digital material on the computer screen. The students can act on the visual diagrams and— through the instrumentation process activated by the tool—they can overcome obstacles and shape their mental schemes, based on schemes of use of the tools/artifacts.

b) On the computer, on the other hand, the student can construct the shapes using the potentialities offered by the software. This means the student can use transformation tools like rotation or reflection in addition to the Compass and Straightedge tool.

According to Whiteley [41] transformations' are the key concept of geometry. Reasoning with transformations should be a central theme of our learning of geometry. In this point we are limited to refer the effects of the construction through rotation. We follow these next steps to create a rotation in Sketchpad v4: to begin with, we select the point which will act as the center for rotation and define it on the transform menu as 'mark center'. Then we select the object we would like to rotate based on an angle, choosing the specified/fixed angle (for example 90^o). When the command runs, a new object is created which is a rotated image of the original object. The rotation of the rectangle for 90 degrees in the software shapes a utilization scheme which leads the students to conceptually grasp the meaning of a) perpendicularity/a right angle; b) congruent shapes. This transformation has a significant impact: during the instrumental approach, the student structures a utilization scheme of the tool, and consequently a mental image of the functional/operational process of rotation, since any modification/ transformation of the initial figure (input) results in the modification/transformation of the final figure (output). [30]

c) The processes can be added to through the construction of custom tools. According to Straesser (see [39]) macros /custom tools "can help to structure a geometrical construction by condensing a complicated sequence of construction steps into one single command".

Straesser [37] points that "…DGS-use can be structured according to conceptual units by means of macro-constructions. DGS-constructions are not bound to follow the small units of traditional drawing practice. Offering new tools that are unavailable in paper and pencil geometry, DGS-use widens the range of accessible geometrical constructions and solutions…[and] provides an access route to deeper reflection and more refined exploration and heuristics than in paper and pencil geometry."

As a result of the construction and application of the custom tool the direct perception of the user is attained with regards to the steps in the development of the construction pertaining to (see [29]): 1) the repetitions in the measurements or calculations of the areas of initial/original shapes 2) the developmental way of the construction of the shape and 3) its orientation towards the sequential steps of the construction on the screen's diagram or in successive pages of the same file. If we save the constructions which refered in the previous subsection as scripts (customs tools in the case of the Geometer's Sketchpad), the problem acquires a different dimension (figure 4.1.2). For example in this case, the student can construct squares and rectangles using the saved tools and two arbitrary segments on the screen (GSP *file page7*[1]). Each shape can now be dragged unmodified on the

[1] **Construction of a "parametrical" square custom tool**

A square can be constructed in Sketchpad using the 'Construction' or 'Transform' menu commands. A square constructed in this way acquires concrete attributes in accordance with the manner of its construction—for example, in relation to the vertex by which it can be dragged and increased or decreased.

We are now going to construct a square which, although it cannot be modified by dragging its vertices, is still a 'dynamic' construction.

We construct a segment of arbitrary length a on the screen. We then construct an arbitrary point C on the screen. Having selected point C and segment a, we construct a circle using the Construct menu and then a radius CD. Selecting points C, D and the segment CD, we construct perpendicular lines.

Is the shape of the square now dynamic? How can we increase or decrease the side of the square now? Try to drag the segment a from its endpoints: what do you observe? The modifications to segment a evoke a corresponding modification to the sides of the screen, while one of the vertexes can be used to change its orientation. This helps in the reconfiguration (see [4]) and construction of a square or, alternatively, the known algebraic expression (proposition 4).(figure 4.1.3) As Rahim [32] writes "mathematically and in a Euclidean sense, this combination allows us to extend the concept of congruence from rigid shapes congruence (same intact shapes) to congruence by pieces (different shapes) thus opening a wider range of interesting possibilities".

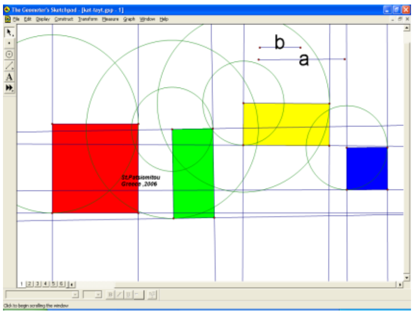

Figure 4.1.1

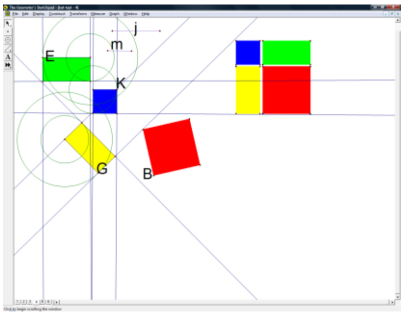

Figure 4.1.2

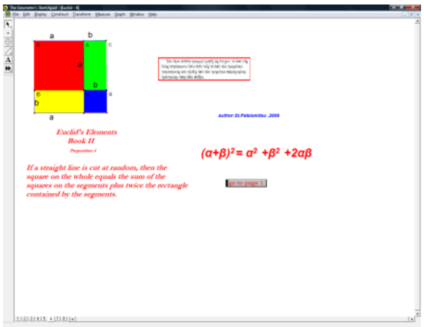

Figure 4.1.3

4. 2 Interaction techniques and transformational processes

Problem 2: *Can a square of side α be transformed into a rectangle if we subtract a square of side b (in one of the corners of the square of side a) from it? If so, can we estimate the area of the shape that remains using only the sides of the two squares, a & b? or Are we able to manipulate the remaining area so that it can be transformated into a rectangle without altering its area?(GSP file page 2)*

In order to subtract the squares we need a visual illustration to the transformation. I semi pre-designed the activities, concentrating on two aspects of the design process: 1) linking the steps in the constructional, transformational or explorative actions or processes in the software using interaction techniques (see [34]) 2) linking the steps in the proof via a sequence of pages or the same page in the DGS environment using interaction techniques. This is what has been defined by the meaning of Linking Visual Active Representations (see for example [30])

The design in the computing environment was inspired by what Freudenthal called "hand-eye performances" [15]. Specifically, the process of proving a problem or theorem is made up of a series of steps which can function as a response, anticipating the questions posed explicitly or implicitly by the teacher (or student). The completion of a step could correspond to a different page in the software connected to the previous page via link action buttons, while on every page different constructional or exploratory actions are linked with different software techniques. The further into the pages we click, the more complex or sophisticated and closer to the problem solving the representation depicted. Thus, a problem would be solved by breaking it down into a series of questions whose answers gradually distil the proof the students seek. The questioning process thus helps students determine and extend their underlying knowledge. Bringing this analysis of the logic

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square. As we noted earlier, a square which depends on the letter a we defined initially on screen is a 'parametrical square', which is to say its sides are modified when segment a is modified, with segment a now acting as a parameter. Now we want to save this construction as a custom tool. We select segment a and the square we have just constructed, defining the new custom tool as a "*parametrical square*", since it can only be modified by segment a. Applying the tool on screen, we construct a linear segment with side x of random/arbitrary length (which we choose), and then a square whose sides is the segment.

of the design process, it should be noted that I bore the following in mind when designing, constructing and implementing the activities: a) the process should be active to keep the students interested and promote discovery; b) students should be guided to reach conclusions. The teacher can guide the students through elucidation or questions to conclusions which compose a step by step visual proof.

Also significant is the role of the software's successive pages, which could be considered as a vivid section of a book revealing the parts of the proof. In this way, the sequence of increasingly sophisticated construction steps in the activity could correspond to the numbering of the action buttons, allowing the student to interact with the tool on his/her own volition or on being encouraged to do so by the teacher during the time allotted for the activity. The link buttons can join different pages and later, more evolved steps in the construction of the representation of the problem. This leads to a cognitive linking of the representations which (see [20]) "creates a whole that is more than the sum of its parts...It enables us to see complex ideas in a new way and apply them more effectively". In that way the pupils can improve their knowledge by having a mental schema elicited from them. As Mariotti in [26] declares "the temporal sequence of the constructions' steps represents the counterpart of the logic hierarchy between the geometric properties of a figure".

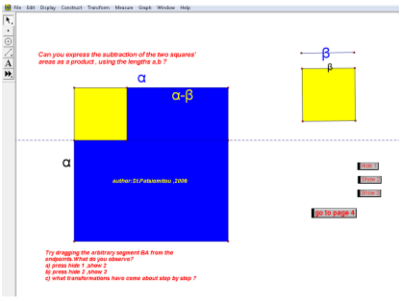

Figure 4.2.1 :step 1 (GSP file page 3)

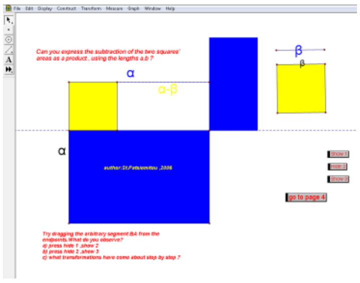

Figure 4.2.2: step 2 and 3 (GSP file page 3)

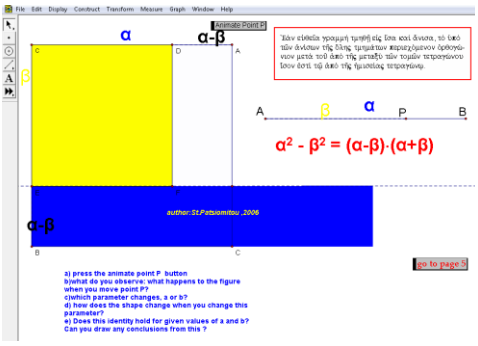

Figure 4.2.3: step 4 (GSP file page 4)

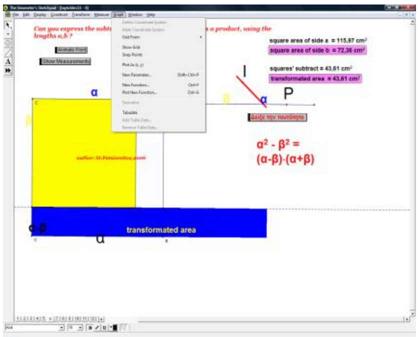

Figure 4.2.4: construction of the red line (GSP file page 4)

dynamic Linking Visual Active Representations (LVAR) of the construction steps

The cognitive linking of the representations could come about through a series of actions which model the problem in the software:

- Step 1) Placing the square of side b on top of the square of side a.(figure 4.2.1)
- Step 2) Rotating the rectangle P_1: this action is connected/completed with a hide/show action button (figure 4.2.2)
- Step 3) Displaying rectangle P_2 vertically by transforming rectangle P_1 by rotating it through 90^o (figure 4.2.2)

- Step 4) Hiding rectangle P_2 and showing rectangle P_3 which resulted from a reflection by line e (figure 4.2.3)

The squares are constructed on the screen using the parametrical mode described above with linear segments ‘a’, ‘b’ which can be modified by manually dragging or using animation. For example the animation process of segment P allows users to observe the corresponding modification: 1) of the subtraction of the two squares’ areas: a^2-b^2 2) the area of rectangle (figure 4.2.3, 4.2.4), which has dimensions a-b, a+b and that they have the same result.

Despite the modification of the segments a, b the empirical visualization of the equality of the areas--the modified shape, and the initial squares areas’ subtraction--can be investigated in the next page of the same file. Segments ‘a’, ‘b’ have been constructed intedependently: segment b belongs on segment a. For this reason, one of the endpoints of segment b (named point P on the diagram)-is located on segment a. Manually dragging point P (or using the animation tool to do so) modifies the length of segment b, and hence both the dependant area of the square and the subtraction between the squares’ areas or in algebraic terms: a^2-b^2 (figures 4.2.3, 4.2.4) The student can observe the alteration of the initial shape as this is made up of equivalent areas and the remaining area as equivalent with a transformated area of a rectangle of dimensions a-b, a+b.

On the other hand, the subtraction of the squares is meaningless if we don’t investigate the relationship between the squares’ sides ‘a’, ‘b’. The activity helps students to form hypotheses and investigate with regard to: 1) The equality or inequality of the segments a, b 2) The relationship between the modification of segment ‘b’ and the modified final area of the rectangle. Dragging segment ‘a’ simultaneously and proportionally modifies segment ‘b’. This leads to the geometrical concept being transformed into an algebraic one, since segments ‘a’, ‘b’ operate as parameters. Consequently the students visual verify that a^2-b^2 = (a-b)(a+b) and that this can be generalized to include any parameters ‘a’, ‘b’. In Fig. 4.2.4, the dynamic link between the parameter and the area of the shape—and thence with the graphic representation leads to the cognitive linking of the parameters in algebra —initially appears the point I as plot of the selected coordinates: of parameter b and of area’s α^2-β^2. Animated point P which is connected with a trace, produces the diverging red line on the screen.

5. Research methodology

The paper represents a quasi-experimental study to utilize *Geometer’s Sketchpad* when promoting relationship between geometric and algebraic expressions. This research sought to investigate the effects of the dynamic geometry software on secondary students’ reasoning and diagrams generation. The qualitative study was conducted in a class at a public high school in Athens, during the second term of the academic year. Twenty eight volunteers 14-15 years old, were randomly divided between the ‘experimental’ and the ‘control’ teams, with 14 students in each. The researcher ensured that both teams consisted of equal number of boys and girls, equal distributed to their achievement in mathematics. The students were friends, which fostered group discussion. The researcher re-formulated problem 1 as follows: Construct 4 shapes: *a square of side x, a square of side 3cm and two rectangles in different colours with sides x, 3cm. Then construct a larger shape matching the shapes which you have already constructed in order to construct a quadrilateral.* The methodology of the class experiment discussed in this paper includes the exploration of the open problems.

Among the students in the control group, identity concepts were initially introduced in the traditional way in class. Namely, the students in the control team had been introduced the meanings using static means, and the students in the experimental team had only worked on the software

constructing the shapes reported in the previous section. Once the students in the experimental group had constructed the shapes in Figure 4.1.1 in the software, a classroom discussion focused on problems 1 & 2. The students in both groups were then asked to construct representations of identities on card as they had understood them through the process, and to represent other algebraic expressions such as trinomials and polynomials.
Consequently, the following research questions can be posed:

1) Can the students in the experimental or control group switch back and forth conceptually between the algebraic expressions and geometric representations?

2) Is there any difference between the two groups in terms of their reasoning and diagrams generation?

5. 1. The control team

The pupils had been introduced by the teacher to the meanings of the identities (the algebraic expressions) in a static way, meaning the teacher proved the identities on the board in the traditional way.
Then the control team was assessed through a paper-pencil test a) regarding the geometric representations of the concrete expressions b) regarding their comprehension of the algebraic expressions and their connection with the areas of geometrical shapes, expecting the students to produce a figure as shown in Fig. 4.1.1-4.1.3. The students were required to notice the dimensions of the shapes and had to try to construct a new quadrilateral (a square). Having accomplished their construction of the square, they had to formulate an algebraic expression which correlated with the area of the new shape. The results of the tests demonstrated the students' inability to compose a larger square matching the shapes which they have already constructed, although they made connections between the algebraic expressions and geometrical representations.

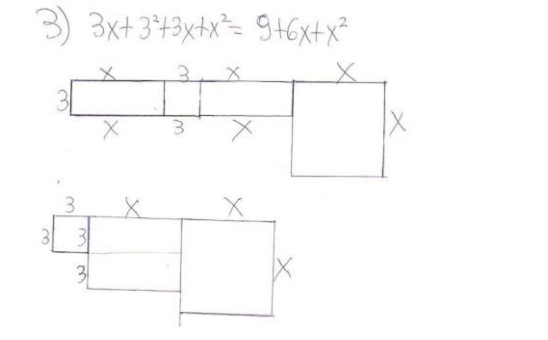

Figure 5.1.1

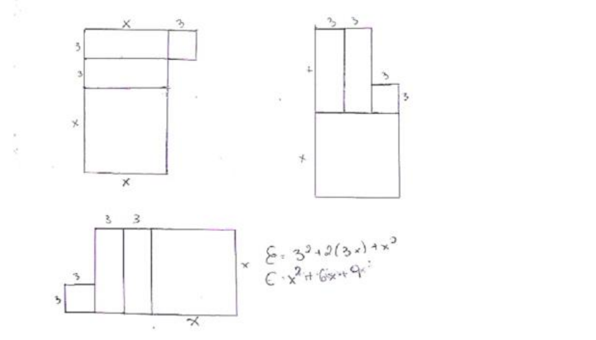

Figure 5.1.2

Students' representations

The shapes in figures 5.1.1, 5.1.2 above have been produced by the most advanced students in the control group. The shapes reveal the students' inability to transform the rectangular shapes in such a way that a larger square is revealed whose sides are equal to x+3. The students have been confused by the variable x and 3cm. Most of the students could not comprehend how formal algebraic expressions could be translated into summations of the areas acquiring dimensions. This is to say they did not understand how an algebraic symbol could be transformed into the area of a two dimensional shape. They encountered difficulties applying the identities through exercises. Another reason they could not use their observations of the shapes to lead them to a square construction, was that most of the students considered the algebraic expressions $(\alpha + \beta)^2$ and $\alpha^2+\beta^2$ to be equal. This was due to cognitive obstacles which had not been overcome through the teaching and learning of the meaning of the identities using static means. This is one of the obstacles the students faced on their way to comprehending the meaning of the equations and then applying them appropriately.

Regarding the problem 2: The researcher had put all the shapes constructed in the software (figures 4.2.1-4.2.4) on a single sheet of paper in the test. The question posed was this*: Can you provide a description and a justification of the correctness of the procedure leading from figure 4.2.1 to figure 4.2.4?*

In problem 2, despite the fact that the shape was given on the test paper and that the pupils had been taught about the axis of symmetry, they failed to notice that the shape in Fig. 4.2.3 were actually the mirror images of the vertical rectangle in Fig. 4.2.2 (in other words how the transformations occurred). Many pupils noticed that the transformed rectangles were congruent; some even made efforts to calculate the lengths of the sides or express them using an abstract parameter, but could not produce the rectangle's actual dimensions, because they had not understood how each shape developed out of the previous step in the construction process. This was a result of a misconception with regard to the evolution of the process on paper in the previous step.

5.2 The experimental team

The session was videotaped and the analysis of the results that will be presented here in the following sections is based on the video and the observations of the researcher during the session. The researcher had hooked a projector to her personal computer and the pupils could participate during the session by

- constructing the shapes in the software using the modes reported above
- interacting in the pre-constructed activity that was already designed in multiple pages.

The current paper seeks to describe the results of this process during the problem solving session rather than the difficulties which occurred during the construction process. In other words, the impact the *structural algebraic units* have on the students' treatment of, and conversion between the different semiotic registers. We will now present the discussion between the students. Four students were predominant in the discourse.

Regarding the re-formulated problem 1:

Iris dragged the shapes on the screen. She observed the shapes and changed their orientation on the screen in order to match them. She finally constructed a square as shown in Figure 4.1.3. (GSP file page 1, 5)

Researcher: *Can you explain how the shapes fit together*?

Alice: *the lengths of the sides are equal and fit together. That's why Iris put the square near the rectangle.*

The students were familiar with software constructions. The procedure led them to note the equality of the segments straight off. Using the circle tool to construct the equal segments led to the construction of a utilization scheme through which the students understood the equalities among the parts of the shape. The custom tools also helped them to understand the shapes' orientation, functioning as instruments.

Iris and Alexandros: *They are equal. The dimensions of the shape are both 3+x.*

Most of the students: *It is a square.*

Filippos: *Its area is* $(3+x)(3+x)$ *namely* $(3+x)^2$

Filippos: *It is a square of side 3, a square of side x and two rectangles of dimensions 3, x*.

Iris and Filippos: *Then* $(3+x)^2$ *is equal to* $3^2+3x+3x+x^2$

She wrote the expression down on her paper.

Filippos: *Now we can understand how it is a mistake to say* $(\alpha+\beta)^2 = \alpha^2 + \beta^2$
Alexandra: *Of course it is!! There are two more terms.*
Filippos: *Because of the two additional areas of the rectangles*

This is a crucial part of the discussion. The students have verified and argued through an epistemological obstacle. Obstacles can be seen as an opportunity for students to reflect on their own learning rather than a barrier to achieve understanding of mathematical ideas. The abstract meanings were rendered concrete by means of the schematic digital material on the computer screen. The students acted on the schemata constructing, composing and—through the instrumentation process activated by the tool—shaping their mental processes.

Regarding problem 2:
The researcher had semi pre-designed the activities in the DGS environment and presented them to the students in the manner described above.
The students pressed the first hide show button.here is an excerpt of their discusion after the first transformation occurred:

Students*: it has been rotated for 90 degrees* (GSP file page 3-press Hide 1)

Then they pressed the next action button

Alexandra: *Now it is altered for 180°*
Iris: *It has been rotated for 270 degrees*. (GSP file page 3-press Hide 1, Hide 2, Show 3)
Filippos: *We had a rotation for 90 degrees and a reflection by this axis of symmetry e .*

The students noticed the transformations, (i.e. the transformations effected using the pre-constructed reflection/rotation commands) which led them to an understanding of the concepts of symmetry and equality between shapes.

He points to the shape.

Researcher: *Can you explain how it can match this side*
Students… *its dimension is equal to a-b* (GSP file page 3-press Hide 1, Hide 2, Show 3)
Students: *… the new shape is a rectangle*.

We calculated the area of the transformed rectangle and the subtraction between the original squares' areas of sides a, b. The students could not comprehend the meaning of "generally". Although students could not grasp the concept of "generally", they did notice the results of the calculations as Filippos pressed the animation button. The rotation of the rectangle for 90 degrees in the software shapes a utilization scheme which leads the students to conceptually grasp the meaning of a) perpendicularity/a right angle; b) congruent shapes.

Filippos: *they're congruent*.
Iris: *what if we change a and b using animation*?
Filippos: *you mean they will change accordingly*.
Students: *so if the values of a and b change, the calculations of the areas will increase or decrease and take many different values.* (GSP file page 4-press Animate Point P)

The students answer in this way purely as a result of having observed the fluctuations in the size of the shapes during the animation process. The students' actions have been shaped by the instrumentation process, so they shape the objects during the instrumentalization process. In the experimental group the procedural knowledge enables the conceptual knowledge development. The instructional implication is: use of procedural knowledge and reflect on the outcome, which in our case is the construction of meanings (see [19]).

5.3 Control group constructions on card

The students in the control group encountered difficulties composing and decomposing the shapes, limiting themselves to constructions relating to those they had already seen in the assessment paper. The shapes in Figs. 5.3.1 and 5.3.3 are representative of how students from the control group went about representing figures 4.2.1-4.2.3 on cardboard.

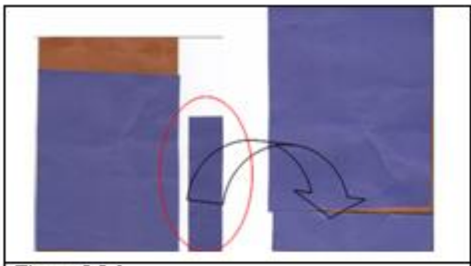

Figure 5.3.1

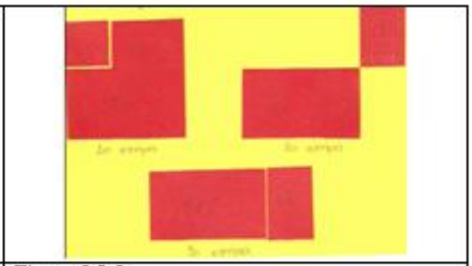

Figure 5.3.2

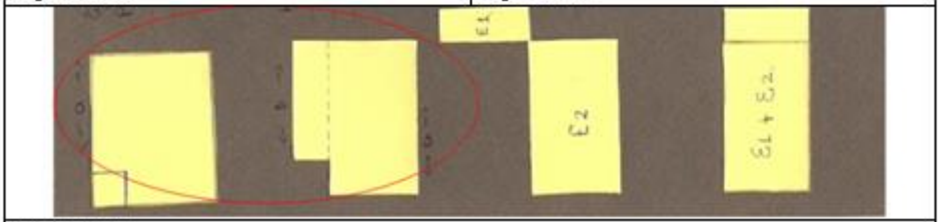

Figure 5.3.3

Coming up against cognitive obstacles they could not overcome, they made mistakes. The static representation of the transformations in the test didn't help them answer with regard to the relationship between the shapes and the way in which the procedure progressed.

The student who produced shape 5.3.1 has noticed that the marked area in the red border must match the one beside the rectangular shape, but failed to construct an initial square in order to work through the procedure. The result was the formation of a rectangle, though not one stemming from the initial shapes. With shapes 5.3.2 & 5.3.3, the students have failed to note the dimensions on their shapes globally (Fig. 5.3.2), or marked them wrongly (Fig. 5.3.3).

5.4 Experimental team constructions on cardbord

The students in the experimental group noted the relationship between area and algebraic expression. The constructions made by the students in the experimental group are far more advanced than those made by the students in the control group.

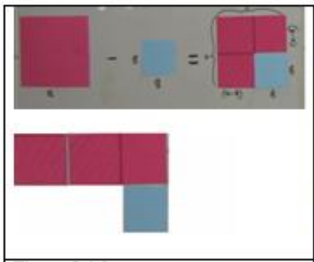

Figure 5.4.1

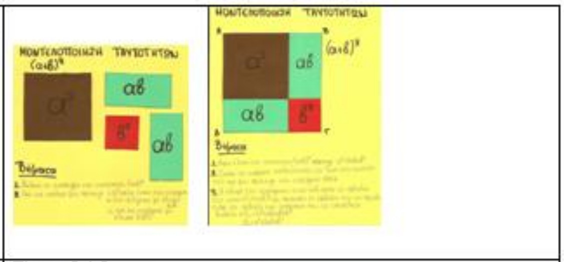

Figure 5.4.2

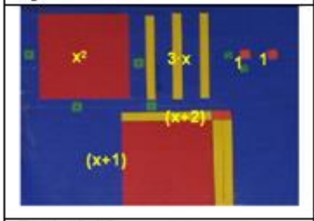

Figure 5.4.3

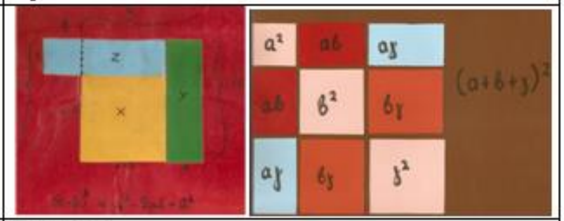

Figure 5.4.4

The students successfully represented identities—trinomials and polynomials—they had not worked on in the software. This is to say that they were able to extend the process. We also note that they assigned dimensions correctly and verified those relating geometric representations with algebraic expressions.

Fig. 5.4.1 was by a student, who did not play an especially active part in the discussion, but here demonstrates that he has understood the way in which we had worked with the software by including the correct dimensions and working out the area of the remaining shape formalistically. Filippos constructed shape in Fig. 5.4.2, which was a representation of the identity a^2+b^2+2ab, Iris constructed the representation of Fig. 5.4.3 ($x^2+ 3x+2$). Alexandra produced shapes in Fig. 5.4.4, which represented the identities $(a-b)^2$ and $(a+b+c)^2$. The students in the experimental group noted the relationship between area and algebraic expression. They switched back and forth conceptually between the algebraic expressions and geometric representations. They also formed an accurate view of the relationship between them and ended up drawing conclusions about Euclid's 4th proposition. In this, they were helped by the constructions of the basic shapes. The constructions in the DGS environment helped them notice the equality of the areas and to apply a puzzle-solving approach. The construction of the action buttons made it easier for them to note the relationship between the sequential linking representations and to react by reflection, noting the equality of the areas and hence, using algebra, the equality of the relationships. We can thus state that the software helped them overcome significant obstacles.

6. Results and Conclusions

The idea of designing a didactical engineering in order to help students bridge algebra and geometry is fundamental in mathematics teaching. The paper provides objective data (e.g the results of the research) to support the idea that working in a DGS environment fulfills this purpose, in other words, that the reported constructions in the software operate as *structural algebraic units* for the students interacting with them, it is obvious that the processes in the software are more efficient, when students are structuring algebraic expressions, than in a paper-pencil-scissors environment. It is necessary to understand the way in which one can form and express with the different semiotic registers the specific mathematical meanings. In the present case we have achieved a transformation on the semiotic registers and for this transform also the semiotic representations. So we have a treatment and a conversion between different semiotic registers. The construction of the action buttons can make easier to note the relationship between the sequential linking representations and to react by reflection, noting the equality of the areas and hence, using algebra, the equality of the relationships.

Dynamic geometry systems have been described as computational environments that embody some subdomain of mathematics or science, generally using linked symbolic and graphical representations. Haapasalo (see [11]) argues that the design of research-based frameworks for instructional praxis needs to take into account mathematics and its history, philosophy, psychology, sociology, physiology, ICT and other fields. Technology has altered our "mental art"—the way we think, plan, work and evaluate in a modern society—holistically.

Sfard (see [35]) claims an interaction between a process or an object is indispensable for a deep understanding of mathematics whatever the definition of "understanding" may be. This process can be repeated, meaning that the level to which one concept is understood can function/operate as a base for developing another. Thus, the tools comprising DGS may well constitute a channel whereby children extend their imagination and conceive mathematics like a source of mathematical models and representations.

Freudenthal in [6] supports that "Socrates did not teach a ready made solution but the way of reinventing the solution." The same approach two millenia later was formulated by Comenius (quoted in [6]): "The best way to teach an activity is to show it." According to Freudenthal "this is a socratic idea, though it involves more than a Socratic lesson." Freudenthal supports that modern educators are likely to subscribe to a variation of Comenius' device while "The best way to teach an activity, is (not) to show it" but rather "The best way to learn an activity, is to perform it."

Overall, when the student uses a dynamic geometry tool and interacts with it, s/he can be encouraged to develop relational conclusions, which in turn helps him/her to develop his geometrical thinking. In other words, when these factors are imposed on logical organization and thinking, conceptual comprehension reaches the level of abstract thought processes as dynamic diagrams can be considered as a metaphor for the corresponding algebraic meanings.

In terms of our historical perspective "this current work is a part of a large transition towards a much more broadly learnable mathematics. These representational innovations …constitute steps toward the development of a new "alphabet" for… mathematics which might do for mathematical representation what the phonetic alphabet did for writing" (see [22]).

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